GEOMETRIC CHARACTERIZATION OF EXCELLENT AFFINE SPHERICAL HOMOGENEOUS SPACES

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Abstract. A spherical homogeneous space of a connected semisimple algebraic group G is said to be excellent if it is quasi-affine and its weight semigroup is generated by disjoint linear combinations of fundamental weights of G. We prove that, for an affine spherical homogeneous space G/H, the condition of being excellent is equivalent to the following two conditions holding simultaneously: first, the factorization morphism $G/H \to Y = \operatorname{Spec} {}^U\mathbb{C}[G/H]$ for the action on G/H of a maximal unipotent subgroup U of G is equidimensional; second, $Y \simeq \mathbb{C}^r$ for some r.

1. Introduction

Let G be a connected semisimple complex algebraic group and U a maximal unipotent subgroup of it. Consider a homogeneous space G/H with finitely generated algebra $\mathbb{C}[G/H] = \mathbb{C}[G]^H$ of regular functions on it. In this situation, the algebra ${}^U\mathbb{C}[G/H]$ consisting of regular functions on G/H that are invariant under the action of U on the left is also finitely generated (see [Had, Theorem 3.1]). Therefore one can consider the corresponding factorization morphism

$$\pi_U \colon X = \operatorname{Spec} \mathbb{C}[G/H] \to Y = \operatorname{Spec} {}^U\mathbb{C}[G/H].$$

The algebra $\mathbb{C}[G/H]$ is a rational G-module with respect to the action of G on the left and decomposes into a direct sum of finite-dimensional irreducible G-modules. The highest weights of irreducible G-modules that occur in this decomposition form a semigroup called the *weight semigroup* of G/H. We denote this semigroup by $\Lambda_+(G/H)$.

A subgroup $H \subset G$ (resp. a homogeneous space G/H) is said to be *spherical* if a Borel subgroup $B \subset G$ has an open orbit in G/H. For a spherical homogeneous space G/H the algebra $\mathbb{C}[G/H]$ is finitely generated [Kno], therefore the morphism π_U is well defined. Further, it is known (see [VK, Theorem 1]) that for a spherical homogeneous space G/H the G-module $\mathbb{C}[G/H]$ is *multiplicity free* (in the case of quasi-affine G/H the converse is also true), that is, every irreducible submodule occurs in this G-module with multiplicity at most 1. In this situation, if we fix a highest weight vector (with respect to U) in each irreducible submodule of $\mathbb{C}[G/H]$, then these vectors form a basis of the algebra ${}^U\mathbb{C}[G/H]$ (regarded as a vector space over $\mathbb{C}$). Moreover, if we normalize these vectors in an appropriate way, we can establish a natural isomorphism between ${}^U\mathbb{C}[G/H]$ and the semigroup algebra of the semigroup $\Lambda_+(G/H)$ (see [Pop, Theorem 2]).

Date: June 29, 2011.

2010 *Mathematics Subject Classification.* 14L30, 14M27, 14M17.

Key words and phrases. Algebraic group, homogeneous space, spherical subgroup, equidimensional morphism, semigroup.

Partially supported by Russian Foundation for Basic Research, grant no. 09-01-00648.

Let $\omega_1, \ldots, \omega_l$ be all the fundamental weights of G. For every dominant weight $\lambda = k_1\omega_1 + \ldots + k_l\omega_l$, where $k_i \in \mathbb{Z}$ and $k_i \geqslant 0$ for all $i = 1, \ldots, l$, we introduce its *support* $\operatorname{Supp}\lambda = \{\omega_i \mid k_i > 0\}$.

A spherical homogeneous space G/H is said to be *excellent* if it is quasi-affine and the semigroup $\Lambda_+(G/H)$ is generated by dominant weights $\lambda_1, \ldots, \lambda_m$ of G with $\operatorname{Supp}\lambda_i \cap \operatorname{Supp}\lambda_j = \varnothing$ for $i \neq j$. For example, for the (affine) spherical homogeneous space $\mathrm{SL}_{2n}/\mathrm{S}(\mathrm{L}_n \times \mathrm{L}_n)$ the weight semigroup is (freely) generated by the weights $\omega_1 + \omega_{2n-1}$, $\omega_2 + \omega_{2n-2}$, …, $\omega_{n-1} + \omega_{n+1}$, $2\omega_n$, where ω_i is the ith fundamental weight of SL_{2n}, hence this space is excellent. It follows from the definition that for an excellent spherical homogeneous space G/H the semigroup $\Lambda_+(G/H)$ is free. Therefore the algebra ${}^U\mathbb{C}[G/H]$ is also free and the variety Y, which is the spectrum of this algebra, is nothing else than the affine space $\mathbb{C}^r$, where r is the rank of $\Lambda_+(G/H)$.

A spherical homogeneous space G/H is said to be *almost excellent*[1] if it is quasi-affine and the convex cone $\mathbb{Q}_+\Lambda_+(G/H)$ consisting of all linear combinations of elements of $\Lambda_+(G/H)$ with non-negative rational coefficients is generated (as a convex cone) by elements $\lambda_1, \ldots, \lambda_m \in \Lambda_+(G/H)$ with $\operatorname{Supp}\lambda_i \cap \operatorname{Supp}\lambda_j = \varnothing$ for $i \neq j$. As can be easily seen (see [Av1, Corollary 1]), for a homogeneous space G/H the property of being almost excellent is local, that is, depends only on the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$.

There is a close connection between excellent and almost excellent spherical homogeneous spaces. This connection can be expressed in the following two facts. First, from the definitions it follows that every excellent spherical homogeneous space is almost excellent. Second, for every almost excellent spherical homogeneous space its simply connected covering homogeneous space is excellent (see [Av1, Theorem 3]).

In 1999 Panyushev proved the following theorem.

Theorem 1. *Let G/H be a quasi-affine spherical homogeneous space. Then:*

(a) [Pa2, Theorem 5.5] *if G/H is excellent, then the morphism π_U is equidimensional;*

(b) [Pa2, Theorem 5.1] *if $Y \simeq \mathbb{C}^r$ for some r and H contains a maximal unipotent subgroup of G, then the converse to* (a) *holds.*

Remark 1. When we say that a morphism is equidimensional we have in mind that it is surjective.

Remark 2. Actually Panyushev proved a more general statement formulated in other terms. The term 'excellent spherical homogeneous space' appeared in 2007 when Theorem 1 was reproved independently by E. B. Vinberg and S. G. Gindikin.

In view of the above-mentioned connection between excellent and almost excellent spherical homogeneous spaces, Theorem 1 implies a similar result for almost excellent spherical homogeneous spaces (see also § 3):

Corollary 1. *Let G/H be a quasi-affine spherical homogeneous space. Then:*

(a) *if G/H is almost excellent, then the morphism π_U is equidimensional;*

(b) *if H contains a maximal unipotent subgroup of G, then the converse to* (a) *holds.*

[1] In the paper [Av1] an erroneous definition of an almost excellent spherical homogeneous space is given. Namely, by mistake in this definition the condition of quasi-affinity of a homogeneous space is missing.

As can be seen, Theorem 1(b) (resp. Corollary 1(b)) is the converse of Theorem 1(a) (resp. of Corollary 1(a)) for spherical homogeneous spaces G/H such that H contains a maximal unipotent subgroup of G (such subgroups H are called *horospherical*).

The goal of the present paper is to establish the converse of Theorem 1(a) and of Corollary 1(a) in the case of *affine* spherical homogeneous spaces G/H, that is, with reductive H. Namely, in this paper we prove the following theorem.

Theorem 2. *Let G/H be an affine spherical homogeneous space. Then:*
(a) *if the morphism π_U is equidimensional, then G/H is almost excellent;*
(b) *if the morphism π_U is equidimensional and $Y \simeq \mathbb{C}^r$ for some r, then G/H is excellent.*

Doubtless, it would be interesting to understand to which extent this result can be generalized to the case of arbitrary quasi-affine spherical homogeneous spaces.

Theorem 1(a), Corollary 1(a), and Theorem 2 imply the following geometric characterization of excellent and almost excellent affine spherical homogeneous spaces.

Corollary 2. *Let G/H be an affine spherical homogeneous space. Then:*
(a) *G/H is almost excellent if and only if the morphism π_U is equidimensional;*
(b) *G/H is excellent if and only if the morphism π_U is equidimensional and $Y \simeq \mathbb{C}^r$ for some r.*

The paper is organized as follows. In § 3 we reformulate Theorem 2 in a form that is more convenient to prove (see Theorem 3). In §§ 4–7 we collect all notions and results needed for the proof of Theorem 3. Namely, in § 4 we recall the classification of affine spherical homogeneous spaces; in § 5 we prove that under some restrictions on a homogeneous space G/H the null fiber of the morphism π_U is non-empty; in § 6 we consider symmetric linear actions of tori; in § 7 we recall the notion of the extended weight semigroup of a homogeneous space. The proof of Theorem 3 is carried out in §§ 8–9. More precisely, in § 8 we reduce the proof of this theorem to the case of strictly irreducible spaces, which, in its turn, is considered in § 9.

The author expresses his gratitude to E. B. Vinberg for discussion of the paper and valuable comments.

2. Some conventions and notation

In this paper the base field is the field $\mathbb{C}$ of complex numbers, all topological terms relate to the Zarisky topology, all groups are assumed to be algebraic and their subgroups closed. The Lie algebras of groups denoted by capital Latin letters are denoted by the corresponding small German letters. For every group L we denote by $\mathfrak{X}(L)$ the character lattice of L.

Throughout the paper, G stands for a connected semisimple algebraic group. We assume a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$ to be fixed. We denote by U the maximal unipotent subgroup of G contained in B. We identify the lattices $\mathfrak{X}(B)$ and $\mathfrak{X}(T)$ by restricting the characters from B to T.

The actions of G on itself by left translation ($(g, x) \mapsto gx$) and right translation ($(g, x) \mapsto xg^{-1}$) induce its representations on the space $\mathbb{C}[G]$ of regular functions on G by the formulae $(gf)(x) = f(g^{-1}x)$ and $(gf)(x) = f(xg)$, respectively. For short, we call them the action *on the left* and the action *on the right*. For every subgroup $L \subset G$ we

denote by ${}^L\mathbb{C}[G]$ (resp. by $\mathbb{C}[G]^L$) the algebra of functions in $\mathbb{C}[G]$ that are invariant under the action of L on the left (resp. on the right).

Two homogeneous spaces G_1/H_1 and G_2/H_2 are said to be *locally isomorphic* if their simply connected covering homogeneous spaces are isomorphic. This is equivalent to the existence of an isomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$ taking $\mathfrak{h}_1$ to $\mathfrak{h}_2$.

Without loss of generality, for every simply connected homogeneous space G/H we assume that G is simply connected and H is connected.

For a group L, the notation $L = L_1 \cdot L_2$ means that L is an *almost direct product* of subgroups $L_1, L_2 \subset L$, that is, $L = L_1L_2$, the subgroups L_1, L_2 elementwise commute and the intersection $L_1 \cap L_2$ is finite.

The list of notation:

e is the identity element of an arbitrary group;

$\mathbb{C}^\times$ is the multiplicative group of $\mathbb{C}$;

V^* is the space of linear functions on a vector space V;

$\Lambda_+(G) \subset \mathfrak{X}(B)$ is the semigroup of dominant weights of G with respect to B;

$V(\lambda)$ is the irreducible G-module with highest weight $\lambda \in \Lambda_+(G)$;

$v_\lambda \in V(\lambda)$ is a highest weight vector in $V(\lambda)$ with respect to B;

λ^* is the highest weight of the irreducible G-module $V(\lambda)^*$;

L^0 is the connected component of the identity of a group L;

L' is the derived subgroup of a group L;

$Z(L)$ is the center of a group L;

$N_L(K)$ is the normalizer of a subgroup K in a group L;

$\operatorname{diag}(a_1, \ldots, a_n)$ is the diagonal matrix of order n with elements $a_1, \ldots, a_n$ on the diagonal.

3. Reformulation of the main theorem

In this section we reduce the proof of Theorem 2 to the proof of the following theorem.

Theorem 3. *Let G/H be a simply connected affine spherical homogeneous space such that the morphism π_U is equidimensional. Then G/H is excellent.*

Let G/H be a simply connected spherical homogeneous space. Every homogeneous space that is locally isomorphic to G/H has the form $G/\widetilde{H}$, where H is a finite extension of H, that is, $\widetilde{H}^0 = H$. Put $\widetilde{Y} = \operatorname{Spec}{}^U\mathbb{C}[G/\widetilde{H}]$ and consider the following commutative diagram:

$$\begin{array}{ccc} G/H & \xrightarrow{\pi_U} & Y \\ \downarrow & & \downarrow \\ G/\widetilde{H} & \xrightarrow[\widetilde{\pi}_U]{} & \widetilde{Y} \end{array}$$

In this diagram, the vertical arrows correspond to the factorization morphisms by the finite group $\widetilde{H}/H$, which implies that the condition of π_U being equidimensional is equivalent to the condition of $\widetilde{\pi}_U$ being equidimensional. In view of this Theorem 3 implies Theorem 2(a). Now let us deduce assertion (b) of Theorem 2 from assertion (a). For that, we note that an almost excellent spherical homogeneous space G/H is excellent if

and only if the semigroup $\Lambda_+(G/H)$ is free. The latter is equivalent to the condition that the algebra ${}^U\mathbb{C}[G/H]$ is free, that is, $Y \simeq \mathbb{C}^r$ for some r.

4. The classification of affine spherical homogeneous spaces

In this section we recall the known classification, up to a local isomorphism, of all affine spherical homogeneous spaces or, equivalently, the classification, up to an isomorphism, of all simply connected affine spherical homogeneous spaces. Before we proceed, let us recall some notions.

A direct product of spherical homogeneous spaces

$$(G_1/H_1) \times (G_2/H_2) = (G_1 \times G_2)/(H_1 \times H_2)$$

is again a spherical homogeneous space. Spaces of this kind, as well as spaces locally isomorphic to them, are said to be *reducible* and all others are said to be *irreducible*. A spherical homogeneous space is said to be *strictly irreducible* if the spherical homogeneous space $G/N_G(H)^0$ is irreducible (see [Vin, § 1.3.6]).

The list, up to a local isomorphism, of all strictly irreducible affine spherical homogeneous spaces G/H is known: in the case of simple G it was obtained in [Krä] and in the case of non-simple semisimple G it was obtained in [Mik] and, independently and by another method, in [Bri]. In its entirety, this list is collected in Tables 1 and 2 in [Av1]. (Although different parts of this list can be found in many other papers including the above-mentioned original papers.) In its final shape, the general procedure of obtaining arbitrary affine spherical homogeneous spaces starting with strictly irreducible ones is given in [Yak][2].

We note that for all simply connected strictly irreducible affine spherical homogeneous spaces G/H the corresponding weight semigroups are known. In the case of simple G these semigroups are computed in [Krä] and in the case of non-simple semisimple G they are computed in [Av2]. (More precisely, in [Av2] the corresponding extended weight semigroups are computed, see their definition in § 7).

We subdivide strictly irreducible affine spherical homogeneous spaces into the following three types:

type I: $\dim Z(H) = 1$, the space G/H' is not spherical;

type II: $\dim Z(H) = 1$, the space G/H' is spherical;

type III: $\dim Z(H) = 0$.

We now describe the general procedure of constructing arbitrary simply connected affine spherical homogeneous spaces starting with strictly irreducible ones. Let G_1/H_1, …, G_n/H_n be simply connected affine spherical homogeneous spaces. We recall that the groups $G_1, \ldots, G_n$ are assumed to be simply connected and the subgroups $H_1, \ldots, H_n$ are assumed to be connected. Having performed a renumbering, if necessary, we may assume that for some p, q, where $0 \leqslant p \leqslant q \leqslant n$, the spaces $G_1/H_1, \ldots, G_p/H_p$ are of type I, the spaces $G_{p+1}/H_{p+1}, \ldots, G_q/H_q$ are of type II, and the spaces $G_{q+1}/H_{q+1}, \ldots, G_n/H_n$ are of type III. We put $G = G_1 \times \ldots \times G_n$, $\widetilde{H} = H_1 \times \ldots \times H_n$. Clearly, $Z(\widetilde{H}) = Z(H_1) \times \ldots \times Z(H_n)$ and $\widetilde{H}' = H_1' \times \ldots \times H_n'$. Further, for all $i = 1, \ldots, n$ we have $\operatorname{rk} \mathfrak{X}(H_i) = \dim Z(H_i)$, whence $\mathfrak{X}(\widetilde{H}) \simeq \mathfrak{X}(H_1) \oplus \ldots \oplus \mathfrak{X}(H_q)$ is a lattice of rank q. We

[2] Originally such a procedure was suggested in [Mik], however it proved to be false. In [Yak] this error is pointed out and a correct version of the procedure is given.

denote by $\chi_1, \ldots, \chi_p$ the images in $\mathfrak{X}(\widetilde{H})$ of basis elements of the lattices $\mathfrak{X}(H_1), \ldots, \mathfrak{X}(H_p)$, respectively.

Let Z be a connected subgroup of $Z(\widetilde{H})^0 = Z(H_1) \times \ldots \times Z(H_q)$. Put $H = Z \cdot (H_1' \times \ldots \times H_n') \subset \widetilde{H}$. Then we can consider the character restriction map $\tau : \mathfrak{X}(\widetilde{H}) \to \mathfrak{X}(H)$.

Theorem 4 ([Yak, Theorem 3, Lemma 5])**.** (a) *The space G/H obtained by the procedure described above is spherical if and only if the characters $\tau(\chi_1), \ldots, \tau(\chi_p)$ are linearly independent in $\mathfrak{X}(H)$;*

(b) *Every simply connected affine spherical homogeneous space is isomorphic to one of the spaces G/H obtained by the procedure described above.*

5. The null fiber of the morphism π_U

The main result of this section is Proposition 1.

It is well known that there is the following isomorphism of $(G \times G)$-modules:

$$\mathbb{C}[G] \simeq \bigoplus_{\lambda \in \Lambda_+(G)} V(\lambda) \otimes V(\lambda^*), \tag{1}$$

where in the left-hand side the group $G \times G$ acts on the left and on the right and in each summand of the right-hand side the left (resp. right) factor of $G \times G$ acts on the left (resp. right) tensor factor. Under this isomorphism, an element $v \otimes \xi \in V(\lambda) \otimes V(\lambda^*)$ corresponds to a function in $\mathbb{C}[G]$ whose value at the element $g \in G$ is $\langle g^{-1}v, \xi \rangle = \langle v, g\xi \rangle$, where $\langle\, \cdot\,, \,\cdot\, \rangle$ is the natural pairing between $V(\lambda)$ and $V(\lambda^*) \simeq V(\lambda)^*$.

As can be easily seen, for a fixed subgroup $H \subset G$ the subspace ${}^U\mathbb{C}[G/H]_\lambda \subset {}^U\mathbb{C}[G/H]$, consisting of all T-semi-invariant functions of weight λ, under isomorphism (1) corresponds to the subspace $\langle v_\lambda \rangle \otimes V(\lambda^*)^H \subset V(\lambda) \otimes V(\lambda^*)$. In particular, from this it follows that for an element $\lambda \in \Lambda_+(G)$ the condition $\lambda \in \Lambda_+(G/H)$ is equivalent to the condition that the subspace $V(\lambda^*)^H \subset V(\lambda^*)$ consisting of all H-invariant vectors is non-trivial.

For an arbitrary homogeneous space G/H we denote by $\mathcal{N}(G/H)$ the subset of G/H defined by vanishing of all functions in ${}^U\mathbb{C}[G/H]$ that are T-semi-invariant of a non-zero weight.

Proposition 1. *Suppose that $G = G_1 \times \ldots \times G_s$, where each of the groups G_i is simple. Let $H \subset G$ be a subgroup such that for every simple component G_i of G the projection of H^0 to G_i is not unipotent. Then the set $\mathcal{N}(G/H) \subset G/H$ is non-empty.*

Proof. For $i = 1, \ldots, s$ we put $T_i = T \cap G_i$ so that T_i is a maximal torus in G_i and $T = T_1 \times \ldots \times T_s$. We identify $\mathfrak{X}(T)$ with a sublattice in $\mathfrak{t}^*$ by taking each character $\chi \in \mathfrak{X}(T)$ to its differential $d\chi \in \mathfrak{t}^*$. We consider the rational subspace $\mathfrak{t}^*_{\mathbb{Q}} = \mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathfrak{t}^*$ and fix an inner product $(\cdot\,, \cdot)$ on it invariant under the Weyl group $W = N_G(T)/T$. By means of this inner product we identify $\mathfrak{t}^*_{\mathbb{Q}}$ with the rational subspace

$$\mathfrak{t}_{\mathbb{Q}} = \{x \in \mathfrak{t} \mid \xi(x) \in \mathbb{Q} \text{ for all } \xi \in \mathfrak{t}^*_{\mathbb{Q}}\} \subset \mathfrak{t}.$$

Replacing H by a conjugate subgroup, we may assume that the subgroup $T_H = (T \cap H)^0 \subset H$ is a maximal torus of H. We note that $\dim_{\mathbb{Q}}(\mathfrak{t}_{\mathbb{Q}} \cap \mathfrak{t}_H) = \dim_{\mathbb{C}} \mathfrak{t}_H$. This together with the hypothesis implies that there exists an element $z \in \mathfrak{t}_{\mathbb{Q}} \cap \mathfrak{t}_H$ whose projection to each of the subspaces $\mathfrak{t}_1, \ldots, \mathfrak{t}_s$ is not zero. The element z, regarded as an

element of $\mathfrak{X}(T)\otimes_{\mathbb{Z}}\mathbb{Q}$, is contained in a Weyl chamber $C \subset \mathfrak{X}(T)\otimes_{\mathbb{Z}}\mathbb{Q}$. Again replacing H by a conjugate subgroup (conjugated by an appropriate element of $N_G(T)$), without loss of generality we may assume that C is the dominant Weyl chamber.

Suppose that $\lambda \in \Lambda_+(G)\backslash\{0\}$. Regard the irreducible G-module $V(\lambda^*)$, a lowest weight vector w_{λ^*} of it, and a T-invariant subspace $V'(\lambda^*)$ complementary to w_{λ^*}. Let ν be a linear function on $V(\lambda^*)$ taking w_{λ^*} to a non-zero value and vanishing on $V'(\lambda^*)$. Then ν is a highest weight vector of the irreducible G-module $V(\lambda^*)^* \simeq V(\lambda)$. Let $f \in {}^U\mathbb{C}[G/H]$ be a T-semi-invariant function of weight λ. Under isomorphism (1) it corresponds to a vector $v_f \in V(\lambda^*)^H$ such that $f(g) = \nu(gv_f)$ for all $g \in G$. Let us show that $v_f \in V'(\lambda^*)$. For that it suffices to check that the vector w_{λ^*} is not T_H-invariant. The latter will be fulfilled if we show that $zw_{\lambda^*} \neq 0$. We have $zw_{\lambda^*} = (z, -(\lambda^*)^*)w_{\lambda^*} = -(z,\lambda)w_{\lambda^*}$. Recall the following well-known fact: for every two fundamental weights ω, ω' of G the inequality $(\omega,\omega') \geqslant 0$ holds, and the equality is attained if and only if ω, ω' are fundamental weights of different simple factors of G. Since $\lambda \neq 0$, $z \in C$, and the projection of z to each of the subspaces $\mathfrak{t}_1,\ldots,\mathfrak{t}_s$ is not zero, in view of the above-mentioned fact we obtain that $(z,\lambda) > 0$, whence $zw_{\lambda^*} \neq 0$ and $v_f \in V'(\lambda^*)$. Therefore $f(eH) = 0$. Since λ and f are arbitrary, we obtain that $eH \in \mathcal{N}(G/H)$. □

Remark 3. Generally speaking, the proof does not imply that $eH \in \mathcal{N}(G/H)$ for every subgroup $H \subset G$ satisfying the hypothesis of the proposition. In the proof the condition $eH \in \mathcal{N}(G/H)$ is achieved due to an appropriate choice of H in its conjugacy class in G.

Let G/H be a homogeneous space such that the algebra $\mathbb{C}[G/H]$ is finitely generated. The *null fiber* of the morphism π_U is the subset of X defined by vanishing of all functions in ${}^U\mathbb{C}[G/H]$ that are T-semi-invariant of a non-zero weight.

Corollary 3. *In the hypothesis of Proposition* 1*, assume that the algebra* $\mathbb{C}[G/H]$ *is finitely generated. Then the null fiber of* π_U *is non-empty and intersects* G/H*.*

6. Symmetric linear actions of tori

Suppose we are given a linear action of a quasi-torus S on a vector space V. We recall that the notation $V/\!/S$ stands for the categorical quotient for the action $S : V$, that is, $V/\!/S = \operatorname{Spec}\mathbb{C}[V]^S$. We also recall that the null fiber (that is, the fiber containing zero) of the factorization morphism $V \to V/\!/S$ is called the null cone.

For every character $\chi \in \mathfrak{X}(S)$ we denote by V_χ the weight subspace in V of weight χ with respect to S. We put $\Phi = \{\chi \in \mathfrak{X}(S) \mid V_\chi \neq 0\}\backslash\{0\} \subset \mathfrak{X}(S)$. Then, evidently, we have $V = V_0 \oplus \bigoplus\limits_{\chi\in\Phi} V_\chi$.

Definition 1. The linear action $S : V$ is said to be *symmetric* if $\dim V_\chi = \dim V_{-\chi}$ for every $\chi \in \Phi$.

Until the end of this subsection, we assume that S is a torus and the action $S : V$ is linear and symmetric.

Put $c = \dim V_0$ and $d = (\dim V - c)/2$. For the action $S : V$ there are (not necessarily different) elements $\chi_1,\ldots,\chi_d \in \mathfrak{X}(S)$ such that there is an S-module isomorphism $V \simeq V_0 \oplus \bigoplus\limits_{i=1}^{d}(\mathbb{C}_{\chi_i} \oplus \mathbb{C}_{-\chi_i})$, where for $\chi \in \mathfrak{X}(S)$ we denote by $\mathbb{C}_\chi$ the one-dimensional S-module on which S acts by the character χ.

Definition 2. The action $S : V$ is said to be *excellent* if the elements $\chi_1, \ldots, \chi_d$ are linearly independent in $\mathfrak{X}(S)$.

The description of linear actions of tori such that the factorization morphism is equidimensional (see [PV, § 8.1]) implies that the action $S : V$ is excellent if and only if the factorization morphism $V \to V/\!/S$ is equidimensional. Further, using the method of supports (see [PV, § 5.4]), it is easy to show that the dimension of the null cone of the action $S : V$ is equal to d. As the dimension of an arbitrary fiber of the morphism $V \to V/\!/S$ does not exceed that of the null cone (see [PV, Corollary 1 from Proposition 5.1]), from what we have said above and the theorem on dimensions of fibers of a dominant morphism it follows the following proposition.

Proposition 2. *We have* $\dim V/\!/S \geqslant c + d$*, and the equality is attained if and only if the action* $S : V$ *is excellent.*

7. The extended weight semigroup of a homogeneous space

Let $H \subset G$ be an arbitrary subgroup. For every character $\chi \in \mathfrak{X}(H)$ we consider the subspace

$$V_\chi = \{f \in \mathbb{C}[G] \mid f(gh) = \chi(h)f(g) \text{ for all } g \in G,\ h \in H\}$$

of the algebra $\mathbb{C}[G]$. It is easy to see that the action on the left of G on the space $\mathbb{C}[G]$ preserves V_χ for every $\chi \in \mathfrak{X}(H)$. All pairs of the form (λ, χ), where $\lambda \in \Lambda_+(G)$, $\chi \in \mathfrak{X}(H)$, such that V_χ contains the irreducible G-module with highest weight λ form a semigroup. This semigroup is said to be the *extended weight semigroup* of the homogeneous space G/H. (A more detailed description see in [Av2, § 1.2] or in [AG, § 1.2].) We denote this semigroup by $\widehat{\Lambda}_+(G/H)$. Since $V_0 = \mathbb{C}[G]^H = \mathbb{C}[G/H]$, we obtain

$$\Lambda_+(G/H) \simeq \{(\lambda, \chi) \in \widehat{\Lambda}_+(G/H) \mid \chi = 0\}. \tag{2}$$

We define the subgroup $H_0 \subset H$ to be the common kernel of all characters of H. This subgroup is normal in H and contains H', therefore the group H/H_0 is commutative (and is thereby a quasi-torus). If H is connected, then H/H_0 is also connected and is thereby a torus. In what follows, we identify the groups $\mathfrak{X}(H)$ and $\mathfrak{X}(H/H_0)$ by means of the natural isomorphism $\mathfrak{X}(H/H_0) \to \mathfrak{X}(H)$. We have $\bigoplus\limits_{\chi \in \mathfrak{X}(H)} V_\chi = \mathbb{C}[G]^{H_0} = \mathbb{C}[G/H_0]$. We note that the action of the group $T \times H/H_0$ determines a grading on ${}^U\mathbb{C}[G/H_0]$ by the semigroup $\widehat{\Lambda}_+(G/H)$ (T acts on the left, H/H_0 acts on the right).

We now turn to the situation when H is a spherical subgroup of G. According to [VK, Theorem 1], the sphericity of H is equivalent to the condition that for every $\chi \in \mathfrak{X}(H)$ the representation of G on the space V_χ is multiplicity free. This implies that the action of $T \times H/H_0$ on the space ${}^U\mathbb{C}[G/H_0]$ is multiplicity free. Further, for a spherical subgroup H the semigroup $\widehat{\Lambda}_+(G/H)$ is free (see [AG, Theorem 2], the case of connected H see also in [Av2, Theorem 1]), hence the algebra ${}^U\mathbb{C}[G/H_0]$ is free and is isomorphic to the semigroup algebra of the semigroup $\widehat{\Lambda}_+(G/H)$. Regard the affine space $Y_0 = \operatorname{Spec} {}^U\mathbb{C}[G/H_0] \simeq \mathbb{C}^n$, where $n = \operatorname{rk} \widehat{\Lambda}_+(G/H)$. We equip Y_0 with a structure of a vector space in such a way that $(T \times H/H_0)$-semi-invariant functions that freely generate the algebra ${}^U\mathbb{C}[G/H_0]$ correspond to the coordinate functions on Y_0. The action on the right of H/H_0 on ${}^U\mathbb{C}[G/H_0]$ naturally corresponds to an action of this subgroup on Y_0.

Lemma 1. *For a spherical subgroup $H \subset G$ the action $H/H_0 : Y_0$ is linear. Moreover, if (λ_1, χ_1), …, (λ_n, χ_n) are all indecomposable elements of the (free) semigroup $\widehat{\Lambda}_+(G/H)$, then there is an H/H_0-module isomorphism $Y_0 \simeq \mathbb{C}_{-\chi_1} \oplus \ldots \oplus \mathbb{C}_{-\chi_n}$.*

Proof. Let $f_1, \ldots, f_n \in {}^U\mathbb{C}[G/H_0]$ be non-zero $(T \times H/H_0)$-semi-invariant functions corresponding to the elements (λ_1, χ_1), …, (λ_n, χ_n) of $\widehat{\Lambda}_+(G/H)$, respectively. These functions freely generate the algebra ${}^U\mathbb{C}[G/H_0]$. Interpreting $f_1, \ldots, f_n$ as coordinate functions on Y_0, we obtain the required result. $\square$

Lemma 2. *For a reductive spherical subgroup $H \subset G$ the linear action $H/H_0 : Y_0$ is symmetric.*

Proof. Using isomorphism (1), it is not hard to show that for $\lambda \in \Lambda_+(G)$ and $\chi \in \mathfrak{X}(H)$ the element (λ, χ) is contained in $\widehat{\Lambda}_+(G/H)$ if and only if the subspace $V(\lambda^*)^{(H)}_\chi \subset V(\lambda^*)$ consisting of H-semi-invariant vectors of weight χ is one-dimensional. Suppose that $(\lambda, \chi) \in \widehat{\Lambda}_+(G/H)$. As H is reductive, we have $V(\lambda^*) = V(\lambda^*)^{(H)}_\chi \oplus W$ for some H-invariant subspace $W \subset V(\lambda^*)$. Let $\xi \in V(\lambda^*)^* \simeq V(\lambda)$ be the linear function on $V(\lambda^*)$ taking a basis vector of $V(\lambda^*)^{(H)}_\chi$ to 1 and vanishing on W. Then ξ is an H-semi-invariant element in $V(\lambda)$ of weight $-\chi$. Therefore $(\lambda^*, -\chi) \in \widehat{\Lambda}_+(G/H)$. Thus the map $(\lambda, \chi) \mapsto (\lambda^*, -\chi)$ is an automorphism of the semigroup $\widehat{\Lambda}_+(G/H)$. In particular, under this automorphism indecomposable elements are taken into indecomposable elements. Hence in view of Lemma 1 we get what was required. $\square$

8. Reduction of the proof of Theorem 3 to the case of strictly irreducible spaces

Suppose that G is simply connected and $H \subset G$ is a connected reductive spherical subgroup. We recall that in § 7 we denoted by H_0 the common kernel of all characters of H and introduced the notation Y_0 for the affine space $\operatorname{Spec} {}^U\mathbb{C}[G/H_0] \simeq \mathbb{C}^n$, where $n = \operatorname{rk} \widehat{\Lambda}_+(G/H)$. We put $X_0 = \operatorname{Spec} \mathbb{C}[G/H_0]$. In our situation, we have $X = G/H$, $X_0 = G/H_0$.

The commutative diagram

(3)
$$\begin{tikzcd} \mathbb{C}[G/H_0] & {}^U\mathbb{C}[G/H_0] \arrow[l] \\ \mathbb{C}[G/H] \arrow[u] & {}^U\mathbb{C}[G/H] \arrow[l] \arrow[u] \arrow[ul] \end{tikzcd}$$

of injective homomorphisms of algebras corresponds to the commutative diagram

(4)
$$\begin{tikzcd} G/H_0 = X_0 \arrow[r, "\varphi_U"] \arrow[d, "\psi_X"'] \arrow[dr, "\psi"] & Y_0 \arrow[d, "\psi_Y"] \\ G/H = X \arrow[r, "\pi_U"'] & Y \end{tikzcd}$$

of dominant morphisms of the respective varieties. We recall that the affine space Y_0 is equiped with a structure of a vector space (see § 7) and that the action $H/H_0 : Y_0$ is linear (Lemma 1) and symmetric (Lemma 2). At that, ψ_Y is nothing else than the factorisation morphism for the action $H/H_0 : Y_0$.

Proposition 3. *If the morphism π_U is equidimensional, then the action $H/H_0 : Y_0$ is excellent.*

Proof. Suppose that π_U is equidimensional. Since the morphism ψ_X is also equidimensional, it follows that the morphism $\psi\colon X_0 \to Y$ is equidimensional. Assume that the action $H/H_0 : Y_0$ is not excellent. Let c and d be the corresponding characteristics of this action (see § 6). In view of Proposition 2 we have $\dim Y > c + d$, whence by the theorem on dimensions of fibers of a dominant morphism the codimension of a generic fiber of ψ is more than $c + d$.

From Theorem 4 it follows that there are simply connected strictly irreducible affine spherical homogeneous spaces $G_1/H_1, \ldots, G_n/H_n$ such that $G = G_1 \times \ldots \times G_n$ and $H'_1 \times \ldots \times H'_n \subset H \subset H_1 \times \ldots \times H_n$. Then $G/H_0 \simeq G_1/H'_1 \times \ldots \times G_n/H'_n$. For every $i = 1, \ldots, n$ we put $U_i = U \cap G_i$, $T_i = T \cap G_i$ and fix a set F_i of the algebra ${}^{U_i}\mathbb{C}[G_i/H'_i] \subset {}^{U}\mathbb{C}[G/H_0]$ consisting of $(T_i \times H_i/H'_i)$-semi-invariant functions that freely generate this algebra. Then $F = F_1 \cup \ldots \cup F_n$ is a set of $(T \times H/H_0)$-semi-invariant functions that freely generate the algebra ${}^{U}\mathbb{C}[G/H_0]$.

Let $\Phi \subset \mathfrak{X}(H/H_0)$ be the set of non-zero weights with respect to H/H_0 of all functions in F. In the space $\mathfrak{X}(H/H_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ we fix a hyperplane h that contains no elements of Φ. We choose one of the half-spaces bounded by h and denote by Φ^+ the set of weights in Φ that are contained in this half-space. We put $\Phi^- = -\Phi^+$ so that $\Phi = \Phi^+ \cup \Phi^-$. Let F^+ (resp. F^0, F^-) be the set of functions in F whose weights with respect to H/H_0 belong to Φ^+ (resp. $\{0\}$, Φ^-). For every $i = 1, \ldots, n$ we also put $F_i^+ = F^+ \cap F_i$, $F_i^0 = F^0 \cap F_i$, $F_i^- = F^- \cap F_i$. Clearly, $c = |F^+| = |F^-|$ $d = |F^0|$. It is not hard to see (see [PV, § 5.4]) that the subset of Y_0 defined by vanishing of all $c + d$ functions in $F^+ \cup F^0$ is contained in the null fiber of ψ_Y. Then the subset $\mathcal{N} \subset X_0$ defined by vanishing of the same functions is contained in the fiber $\psi^{-1}(\psi_Y(0))$ of ψ (below we refer to this fiber as the null fiber as well). Let us show that $\mathcal{N} \neq \varnothing$. As can be easily seen, for that it suffices to show that the subset of G_i/H'_i defined by vanishing of all functions in $F_i^+ \cup F_i^0$ is non-empty for every $i = 1, \ldots, n$. To do that, we consider two possibilities.

(1) The group H'_i is trivial. Then H_i is a torus. Having inspected the list of all simply connected strictly irreducible affine spherical homogeneous spaces we find that this is only possible for $G_i = \mathrm{SL}_2$, $H_i \simeq \mathbb{C}^\times$. In this case, F_i contains two functions having non-zero opposite weights with respect to the torus H_i/H'_i. As $\mathrm{SL}_2/\mathbb{C}^\times$ is a strictly irreducible spherical homogeneous space of type I, by Theorem 4(a) the images of these weights in $\mathfrak{X}(H/H_0)$ are also non-zero. Hence $|F_i^+| = |F_i^-| = 1$ and $F_i = F_i^+ \cup F_i^-$. A direct check shows that the subset of SL_2 defined by vanishing of the unique function in F_i^+ is non-empty.

(2) The group H'_i is non-trivial. Having inspected the list of all simply connected strictly irreducible affine spherical homogeneous spaces we find that in this case the group H'_i satisfies the hypothesis of Proposition 1, hence even the subset of G_i/H'_i defined by vanishing of all functions in F_i is non-empty.

Thus, $\mathcal{N} \neq \varnothing$. Since $\mathcal{N}$ is contained in the null fiber of ψ and is defined by vanishing of $c + d$ functions, the codimension in X_0 of the null fiber of ψ is at most $c + d$ and therefore is strictly less than the codimension of a generic fiber. Therefore the morphism ψ is not equidimensional, a contradiction. $\square$

Proposition 4. *Let G/H be a simply connected affine spherical homogeneous space such that the morphism π_U is equidimensional. Then G/H is a direct product of several (simply connected) strictly irreducible affine spherical homogeneous spaces.*

Proof. By Theorem 4 there are simply connected strictly irreducible affine spherical homogeneous spaces G_1/H_1, ..., G_n/H_n such that:

(1) $G = G_1 \times \ldots \times G_n$;

(2) $H_1' \times \ldots \times H_n' \subset H \subset H_1 \times \ldots \times H_n$.

Without loss of generality we may assume that the space G_i/H_i is of type I or II for $i \leqslant q$ and is of type III for $i > q$. For every $i = 1, \ldots, n$ we put $U_i = G_i \cap U$.

Let $\tau\colon \mathfrak{X}(H_1 \times \ldots \times H_n) \to \mathfrak{X}(H)$ be the character restriction map. For $i = 1, \ldots, q$ we denote by χ_i a basis character in $\mathfrak{X}(H_i) = \mathfrak{X}(H_i/H_i')$. Having performed a renumbering, if necessary, we may assume that $\tau(\chi_i) \neq 0$ for $i \leqslant p$ and $\tau(\chi_i) = 0$ for $p < i \leqslant q$. As for every $i = 1, \ldots, p$ the (linear) action of the torus H_i/H_i' on the affine space $\operatorname{Spec}^{U_i}\mathbb{C}[G_i/H_i']$ is non-trivial, from Proposition 3 it follows that the weights $\tau(\chi_1)$, ..., $\tau(\chi_p)$ are pairwise different and linearly independent in $\mathfrak{X}(H)$. This implies that $H = H_1 \times \ldots \times H_p \times H_{p+1}' \times \ldots \times H_n'$ and $G/H \simeq G_1/H_1 \times \ldots \times G_p/H_p \times G_{p+1}/H_{p+1}' \times \ldots \times G_n/H_n'$. □

9. The case of strictly irreducible spaces

In view of Proposition 4, to complete the proof of Theorem 3 it suffices to prove the following proposition.

Proposition 5. *Let G/H be a simply connected strictly irreducible affine spherical homogeneous space that is not excellent. Then the morphism π_U is not equidimensional.*

Proof. Since we have at our disposal the classification (more precisely, the complete list) of simply connected strictly irreducible affine spherical homogeneous spaces (see § 4), it suffices to consider case-by-case all such spaces that are not excellent and to prove the statement by a direct check. We recall that the list, up to an isomorphism, of all simply connected strictly irreducible affine spherical homogeneous spaces is collected in Tables 1 and 2 in [Av1]. In addition, in these tables it is indicated for each of the spaces whether it is excellent or not. Hence we obtain a list of spaces that are needed to be considered for proving the proposition. Before we proceed to this list, let us introduce additional conventions and notation.

If G is a product of several factors, then by π_i, φ_i, and ψ_i we denote the ith fundamental weight of the first, second, and third factor, respectively. At that, the enumeration of fundamental weights of simple groups is the same as in the book [OV].

We denote by E_m the identity matrix of order m and by F_m the matrix of order m with ones on the antidiagonal and zeros elsewhere.

For every matrix denoted by a capital letter, the corresponding small letter with a double index ij denotes the element in the ith row and jth column of this matrix. For example, P is a matrix and p_{ij} is the element in the ith row and jth column of P.

The basis $e_1, \ldots, e_n$ of the space of the tautological linear representation of the group SO_n is assumed to be chosen in such a way that the matrix of the invariant non-degenerate symmetric bilinear form is F_n. The basis $e_1, \ldots, e_{2m}$ of the space of the tautological linear representation of the group Sp_{2m} is assumed to be chosen in such a way that the invariant

non-degenerate skew-symmetric bilinear form has the matrix

$$\begin{pmatrix} 0 & F_m \\ -F_m & 0 \end{pmatrix}.$$

With these choices of bases we may (and shall) assume that for every simple factor $\widetilde{G}$ of G the groups $B \cap \widetilde{G}$, $U \cap \widetilde{G}$, and $T \cap \widetilde{G}$ are represented by upper-triangular, upper unitrianglular, and diagonal matrices.

We now proceed to a consideration of all simply connected strictly irreducible affine spherical homogeneous spaces that are not excellent. We divide the consideration into two cases in dependence of $\dim Z(H)$.

Case 1. $\dim Z(H) = 1$. In this case, for every space G/H under consideration we indicate the indecomposable elements of the semigroup $\widehat{\Lambda}_+(G/H)$. By Lemma 1 these elements completely determine the structure of an H/H_0-module on Y_0. For every space G/H it is easy to check that the action $H/H_0 : Y_0$ is not excellent, whence by Proposition 3 the morphism π_U is not equidimensional.

1°. $G = \mathrm{SL}_{2n+1}$, $H = \mathbb{C}^\times \times \mathrm{Sp}_{2n}$, $n \geqslant 2$. The factor Sp_{2n} of H embedded in G as the upper left $2n \times 2n$ block, and the torus $\mathbb{C}^\times$ is embedded in G by means of the map $s \mapsto \mathrm{diag}(s, \ldots, s, s^{-2n})$.

The information contained in rows 6 and 7 of Table 1 in [Av1] allows us to conclude that the semigroup $\widehat{\Lambda}_+(G/H)$ is freely generated by the elements $(\pi_1, n\chi)$, $(\pi_2, -\chi)$, $(\pi_3, (n-1)\chi)$, $(\pi_4, -2\chi)$, $\ldots$, (π_{2n-1}, χ), $(\pi_{2n}, -n\chi)$ for some non-zero character $\chi \in \mathfrak{X}(H)$. As $n \geqslant 2$, this implies that the action $H/H_0 : Y_0$ is not excellent and the morphism π_U is not equidimensional.

2°. $G = \mathrm{Spin}_{10}$, $H = \mathbb{C}^\times \times \mathrm{Spin}_7$. This homogeneous space is uniquely determined by the locally isomorphic space $\widetilde{G}/\widetilde{H}$, where $\widetilde{G} = \mathrm{SO}_{10}$, $\widetilde{H} = \mathbb{C}^\times \times \mathrm{Spin}_7$. The factor Spin_7 of $\widetilde{H}$ is regarded as a subgroup of SO_8 (the embedding $\mathrm{Spin}_7 \hookrightarrow \mathrm{SO}_8$ is given by the spinor representation), and the group SO_8 is embedded in $\widetilde{G}$ as the central 8×8 block. The torus $\mathbb{C}^\times$ is embedded in $\widetilde{G}$ by means of the map $s \mapsto \mathrm{diag}(s, 1, 1, \ldots, 1, s^{-1})$.

The information contained in row 16 of Table 1 in [Av1] and in row 13 of Table 1 in [Pa1] allows us to conclude that the semigroup $\widehat{\Lambda}_+(G/H)$ is freely generated by the elements $(\pi_1, 2\chi)$, $(\pi_1, -2\chi)$, $(\pi_2, 0)$, (π_4, χ), $(\pi_5, -\chi)$ for some non-zero character $\chi \in \mathfrak{X}(H)$. This implies that the action $H/H_0 : Y_0$ is not excellent and the morphism π_U is not equidimensional.

3°. $G = \mathrm{SL}_n \times \mathrm{SL}_{n+1}$, $H = \mathrm{SL}_n \times \mathbb{C}^\times$, $n \geqslant 2$. The factor SL_n of H is diagonally embedded in G, as the upper left $n \times n$ block in the factor SL_{n+1}. The torus $\mathbb{C}^\times$ is embedded in the factor SL_{n+1} of G by means of the map $s \mapsto \mathrm{diag}(s, \ldots, s, s^{-n})$.

It is indicated in row 1 of Table 1 in [Av2] that the semigroup $\widehat{\Lambda}_+(G/H)$ is freely generated by the elements $(\varphi_1, n\chi)$, $(\pi_{n-1}+\varphi_2, (n-1)\chi)$, $\ldots$, $(\pi_1+\varphi_n, \chi)$, $(\pi_{n-1}+\varphi_1, -\chi)$, $\ldots$, $(\pi_1+\varphi_{n-1}, -(n-1)\chi)$, $(\varphi_n, -n\chi)$ for some non-zero character $\chi \in \mathfrak{X}(H)$. This implies that the action $H/H_0 : Y_0$ is not excellent and the morphism π_U is not equidimensional.

4°. $G = \mathrm{SL}_n \times \mathrm{Sp}_{2m}$, $H = \mathbb{C}^\times \cdot \mathrm{SL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$, $n \geqslant 3$, $m \geqslant 1$. The embedding of H in G is as follows. The factor SL_{n-2} is embedded in the factor SL_n of G as the upper left $(n-2) \times (n-2)$ block. The factor SL_2 is diagonally embedded in G as the lower right 2×2 block in SL_n and as the block 2×2 corresponding to the first and last rows and columns in the factor Sp_{2m}. The factor Sp_{2m-2} is embedded in Sp_{2m} as the central

$(2m-2)\times(2m-2)$ block. At last, the torus $\mathbb{C}^\times$ embedded in the factor SL_n of G by means of the map $s\mapsto \mathrm{diag}(s^{-2},\ldots,s^{-2},s^{n-2},s^{n-2})$ for odd n and by means of the map $s\mapsto \mathrm{diag}(s^{-1},\ldots,s^{-1},s^{\frac{n-2}{2}},s^{\frac{n-2}{2}})$ for even n.

It is indicated in row 3 of Table 1 in [Av2] that the semigroup $\widehat{\Lambda}_+(G/H)$ is freely generated by the elements $(\pi_{n-2},2\chi)$, $(\varphi_2,0)$ (this element is contained in the set of indecomposable elements for $m\geqslant 2$), $(\pi_{n-1}+\varphi_1,\chi)$, $(\pi_1+\pi_{n-1},0)$ (this elements is contained in the set of indecomposable elements for $n\geqslant 4$), $(\pi_1+\varphi_1,-\chi)$, $(\pi_2,-2\chi)$ for some nonzero character $\chi\in\mathfrak{X}(H)$. This implies that the action $H/H_0 : Y_0$ is not excellent and the morphism π_U is not equidimensional.

Case 2. $\dim Z(H)=0$. For each of the spaces G/H below we fix and denote by F a set of T-semi-invariant functions that freely generate the algebra ${}^U\mathbb{C}[G/H]$. The weights of these functions are the indecomposable elements of the (free) semigroup $\Lambda_+(G/H)\simeq\widehat{\Lambda}_+(G/H)$. We put $|F|=r$. In all the cases the null fiber of the morphism π_U is non-empty by Proposition 1. To consider the spaces of series 2° and 4° we shall need the following auxiliary lemma.

Lemma 3. *Let $\gamma\colon M\to N$ be a morphism of affine algebraic varieties. Suppose that a closed subvariety $N_0\subset N$ is such that the set $\gamma(M)\cap N_0$ is dense in N_0. Then $\mathrm{codim}_M\,\gamma^{-1}(N_0)\leqslant\mathrm{codim}_N\,N_0$.*

Proof. Let $\widetilde{N}$ denote the closure of $\gamma(M)$ in N. As the morphism $\gamma\colon M\to\widetilde{N}$ is dominant, the dimension of its generic fiber equals $\dim M-\dim\widetilde{N}$. Further, the restriction of γ to the subvariety $\gamma^{-1}(N_0)$ is also a dominant morphism, therefore the dimension of the fiber of a generic point in N_0 equals $\dim\gamma^{-1}(N_0)-\dim N_0$. Hence $\dim\gamma^{-1}(N_0)-\dim N_0\geqslant\dim M-\dim\widetilde{N}\geqslant\dim M-\dim N$, that is, $\dim M-\dim\gamma^{-1}(N_0)\leqslant\dim N-\dim N_0$. □

1°. $G=\mathrm{Spin}_n\times\mathrm{Spin}_{n+1}$, $H=\mathrm{Spin}_n$, $n\geqslant 3$. The homogeneous space G/H is uniquely determined by the locally isomorphic homogeneous space $\widetilde{G}/\widetilde{H}$, where $\widetilde{G}=\mathrm{SO}_n\times\mathrm{SO}_{n+1}$, $\widetilde{H}=\mathrm{SO}_n$, and the subgroup $\widetilde{H}$ is diagonally embedded in $\widetilde{G}$. Let $\theta_m:\mathrm{SO}_m\hookrightarrow\mathrm{SO}_{m+1}$ be the embedding induced by the embedding $\mathbb{C}^m\hookrightarrow\mathbb{C}^{m+1}$ sending the basis $(e_1,\ldots,e_m)$ to the basis $(e_1,\ldots,e_{\frac{m}{2}},e_{\frac{m}{2}+2},\ldots e_{m+1})$ for even m and to the basis $(e_1,\ldots,e_{\frac{m-1}{2}},\frac{1}{\sqrt{2}}(e_{\frac{m+3}{2}}+e_{\frac{m+1}{2}}),e_{\frac{m+5}{2}},\ldots,e_{m+1})$ for even m. We fix the embedding of $\widetilde{H}$ in $\widetilde{G}$ such that the image in $\widetilde{G}$ of the matrix $P\in\widetilde{H}$ is $(P,\theta_n(P))$. The covering homomorphism $G/H\to\widetilde{G}/\widetilde{H}$ determines the natural embedding $\mathbb{C}[\widetilde{G}/\widetilde{H}]\hookrightarrow\mathbb{C}[G/H]$. In view of this embedding every regular function on $\widetilde{G}/\widetilde{H}$ will be considered also as a regular function on G/H.

For $n=3$ the homogeneous space G/H is isomorphic to the space $(\mathrm{SL}_2\times\mathrm{SL}_2\times\mathrm{SL}_2)/\mathrm{SL}_2$, where the sugroup SL_2 is diagonally embedded in $\mathrm{SL}_2\times\mathrm{SL}_2\times\mathrm{SL}_2$. This space is isomorphic to the space of series 4° (see below) with $n=m=l=1$. For $n=4$ the homogeneous space G/H is isomorphic to the space of series 5° (see below) with $n=m=1$. Further we assume that $n\geqslant 5$.

Suppose that $g=(P,Q)\in\widetilde{G}$. Put $R=Q\theta_n(P)^{-1}$. For $n=2k$ the set F contains functions that are proportional to the following functions of g: $f_1=r_{n+1,1}$, $f_2=r_{n+1,k+1}$, and $f_3=f_1r_{n,k+1}-f_2r_{n,1}$; for $n=2k+1$ the set F contains functions that are proportional to the following functions of g: $f_1=r_{n+1,1}$, $f_2=r_{n+1,k+1}-r_{n+1,k+2}$, and $f_3=f_1(r_{n,k+1}-r_{n,k+2})-f_2r_{n,1}$ (see [Av2, § 3.1, Case 2]). Further without loss of generality we assume

that $f_1, f_2, f_3 \in F$. The weights with respect to T of the functions f_1, f_2 are $\pi_1 + \varphi_1$, φ_1, respectively, and the weight of f_3 is $\pi_1 + \varphi_2$ for $n \geqslant 6$ and $\pi_1 + \varphi_2 + \varphi_3$ for $n = 5$. Since the condition $f_1 = f_2 = 0$ implies that $f_3 = 0$, the subset of G/H defined by vanishing of all functions in F coincides with the subset of G/H defined by vanishing of all functions in $F \backslash \{f_3\}$. Thus the codimension of the null fiber of the morphism π_U is at most $r-1$, whence π_U is not equidimensional.

2°. $G = \mathrm{SL}_n \times \mathrm{Sp}_{2m}$, $H = \mathrm{SL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$, $n \geqslant 5$, $m \geqslant 1$. The factor SL_{n-2} of H is embedded in the factor SL_n of G as the upper left $(n-2)\times(n-2)$ block. The factor SL_2 of H is diagonally embedded in G as the lower right 2×2 block in SL_n and as the 2×2 block corresponding to the first and last rows and columns in the factor Sp_{2m}. The factor Sp_{2m-2} of H is embedded in the factor Sp_{2m} of G as the central $(2m-2)\times(2m-2)$ block.

Suppose that $g = (P, Q) \in G$. We denote by P_{ij} the (i, j)-cofactor of the matrix P. The set F contains functions proportional to the following functions of g: $f_1 = p_{n,n-1}q_{2m,2m} - p_{n,n}q_{2m,1}$, $f_2 = p_{n,n-1}P_{1,n-1} + p_{nn}P_{1,n}$, $f_3 = q_{2m,1}P_{1,n-1} + q_{2m,2m}P_{1,n}$ (see [Av2, § 3.2, Case 4]). The weights with respect to T of the functions f_1, f_2, f_3 are $\pi_{n-1}+\varphi_1$, $\pi_1+\pi_{n-1}$, $\pi_1+\varphi_1$, respectively. Let $\gamma\colon G/H \to \mathbb{C}^6$ be the morphism defined by the functions $P_{1,n-1}$, $P_{1,n}$, $p_{n,n-1}$, $p_{n,n}$, $q_{2m,1}$, $q_{2m,2m}$. Let N_0 denote the subset of $\mathbb{C}^6$ defined by $f_1 = f_2 = f_3 = 0$. As can be easily seen, $\dim N_0 = 4$. Now, given non-zero numbers a, b, c, d we consider the pair of matrices

$$P_0 = \begin{pmatrix} 0 & \dots & 0 & b^{-1} \\ \vdots & E_{n-3} & \vdots & \vdots \\ d^{-1} & \dots & 0 & 0 \\ 0 & \dots & bd & -ad \end{pmatrix}, \quad Q_0 = \begin{pmatrix} a^{-1}c^{-1} & \dots & 0 \\ \vdots & E_{2m-2} & \vdots \\ -bc & \dots & ac \end{pmatrix}$$

(the dots stand for zero entries). We have $(P_0, Q_0) \in G$ for every non-zero values of a, b, c, d, at that, the values of the functions $P_{1,n-1}$, $P_{1,n}$, $p_{n,n-1}$, $p_{n,n}$, $q_{2m,1}$, $q_{2m,2m}$ on the pair (P_0, Q_0) are a, b, bd, $-ad$, $-bc$, ac, respectively. Besides, it is easy to check that $\gamma((P_0, Q_0)H) \in N_0$. This implies that $\dim(\gamma(G/H) \cap N_0) = 4$, that is, the set $\gamma(G/H) \cap N_0$ is dense in N_0. Then by Lemma 3 the codimension in G/H of the subset defined by vanishing of the functions f_1, f_2, f_3 is at most 2. Hence the codimension in G/H of the subset defined by vanishing of all functions in F is at most $r-1$. Therefore the morphism π_U is not equidimensional.

3°. $G = \mathrm{Sp}_{2n} \times \mathrm{Sp}_4$, $H = \mathrm{Sp}_{2n-4} \times \mathrm{Sp}_4$, $n \geqslant 3$. The first factor of H is embedded in the first factor of G as the central $(2n-4)\times(2n-4)$ block; the second factor of H is diagonally embedded in G, as the 4×4 block in rows and columns 1, 2, $2n-1$, $2n$ in the first factor.

Suppose that $g = (P, Q) \in G$. We put $R = PQ^{-1} \in \mathrm{Sp}_{2n}$ (the matrix Q is embedded in Sp_{2n} as the 4×4 block in rows and columns 1, 2, $2n-1$, $2n$). The set F contains functions that are proportional to the following functions of g: $f_1 = r_{2n,1}$, f_2 is the minor of order 3 of R corresponding to the last three rows and columns 1, 2, $2n-1$, and $f_3 = f_1W + f_2r_{2n,2}$, where W is the minor of order 3 of R corresponding to the last three rows and columns 1, 2, n (see [Av2, § 3.2, Case 6]). Further without loss of generality we assume that $f_1, f_2, f_3 \in F$. The weights with respect to T of the functions f_1, f_2, f_3 are $\pi_1 + \varphi_1$, $\pi_3 + \varphi_1$, $\pi_1 + \pi_3 + \varphi_2$, respectively. As the condition $f_1 = f_2 = 0$ implies that

$f_3 = 0$, the subset of G/H defined by vanishing of all functions in F coincides with the subset of G/H defined by vanishing of all functions in $F\backslash\{f_3\}$. Thus the codimension of the null fiber of the morphism π_U is at most $r-1$, hence π_U is not equidimensional.

4°. $G = \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \times \mathrm{Sp}_{2l}$, $H = \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_{2m-2} \times \mathrm{Sp}_{2l-2} \times \mathrm{Sp}_2$, $n, m, l \geqslant 1$. Each of the first three factors of H is embedded in the respective factor of G as the central block of the corresponding size. The factor Sp_2 of H is diagonally embedded in G as the 2×2 block corresponding to the first and last rows and columns in each factor.

Suppose that $g = (P, Q, R) \in G$. The set F contains functions proportional to the following functions of g: $f_1 = p_{2n,1}q_{2m,2m} - p_{2n,2n}q_{2m,1}$, $f_2 = q_{2m,1}r_{2l,2l} - q_{2m,2m}r_{2l,1}$, and $f_3 = p_{2n,1}r_{2l,2l} - p_{2n,2n}r_{2l,1}$ (see [Av2, § 3.2, Case 7]). The weights with respect to T of the functions f_1, f_2, f_3 are $\pi_1 + \varphi_1$, $\varphi_1 + \psi_1$, $\pi_1 + \psi_1$, respectively. Let $\gamma\colon G/H \to \mathbb{C}^6$ be the morphism defined by the functions $p_{2n,1}$, $p_{2n,2n}$, $q_{2m,1}$, $q_{2m,2m}$, $r_{2l,1}$, $r_{2l,2l}$. Let N_0 denote the subset in $\mathbb{C}^6$ defined by $f_1 = f_2 = f_3 = 0$. As can be easily seen, $\dim N_0 = 4$. Each of the functions $p_{2n,1}$, $p_{2n,2n}$, $q_{2m,1}$, $q_{2m,2m}$, $r_{2l,1}$, $r_{2l,2l}$ is semi-invariant with respect to the action of the group $T \times T$, where the left factor acts on the left and the right factor acts on the right. At that, the weights of all these functions are linearly independent in $\mathfrak{X}(T \times T)$. This implies that for every set of non-zero numbers a, b, c, d there is a triple of matrices $(P_0, Q_0, R_0) \in G$ such that the values of the functions $p_{2n,1}$, $p_{2n,2n}$, $q_{2m,1}$, $q_{2m,2m}$, $r_{2l,1}$, $r_{2l,2l}$ on (P_0, Q_0, R_0) are a, ad, b, bd, c, cd, respectively. Besides, it is easy to check that $\gamma((P_0, Q_0, R_0)H) \in N_0$ for any non-zero values of a, b, c, d. This implies that $\dim(\gamma(G/H) \cap N_0) = 4$, that is, the set $\gamma(G/H) \cap N_0$ is dense in N_0. Then by Lemma 3 the codimension in G/H of the subset defined by vanishing of the functions f_1, f_2, f_3 is at most 2. Therefore the codimension in G/H of the set defined by vanishing of all functions in F is at most $r-1$. Hence the morphism π_U is not equidimensional.

5°. $G = \mathrm{Sp}_{2n} \times \mathrm{Sp}_4 \times \mathrm{Sp}_{2m}$, $H = \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2m-2}$, $n, m \geqslant 1$. The first factor of H is embedded in the first factor of G as the central $(2n-2) \times (2n-2)$ block. The fourth factor of H is similarly embedded in the third factor of G. The second factor of H is diagonally embedded in the first and second factors of G as the 2×2 block in the first and last rows and columns. The third factor of H is diagonally embedded in the second and third factors of G as the central 2×2 block in the second factor and as the 2×2 block in the first and last rows and columns in the third factor.

Suppose that $g = (P, Q, R) \in G$. The set F contains functions that are proportional to the following functions of g: $f_1 = p_{2n,1}q_{44} - p_{2n,2n}q_{41}$, $f_2 = r_{2m,1}q_{43} - r_{2m,2m}q_{42}$, and $f_3 = f_2(p_{2n,1}q_{34} - p_{2n,2n}q_{31}) - f_1(r_{2m,1}q_{33} - r_{2m,2m}q_{32})$ (see [Av2, § 3.2, Case 8]). Further without loss of generality we assume that $f_1, f_2, f_3 \in F$. The weights with respect to T of the functions f_1, f_2, f_3 are $\pi_1 + \varphi_1$, $\varphi_1 + \psi_1$, $\pi_1 + \varphi_2 + \psi_1$, respectively. As the condition $f_1 = f_2 = 0$ implies that $f_3 = 0$, the subset of G/H defined by vanishing of all functions in F coincides with the subset of G/H defined by vanishing of all functions in $F\backslash\{f_3\}$. Thus the codimension of the null fiber of the morphism π_U is at most $r-1$, therefore π_U is not equidimensional. □

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